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Abstract. We compute the normalisation matrix and Yukawa couplings on the Z-manifold in the geometric approximation where the singularities on the Z-orbifold are blown up with Eguchi-Hanson metrics. The orbifold limit is discussed in detail from a geometric and algebraic point of view, and the results are compared with those obtained using algebraic geometry and conformal field theory.

1. Introduction

Current candidate ground states of superstrings are of the type $M^4 \times K^6$ where M^4 is a maximally symmetric four-dimensional spacetime and K^6 is either a six-dimensional manifold of the Calabi-Yau type [1], or a singular object called an 'orbifold' by physicists [2].

A large number of Calabi-Yau manifolds are known [1, 3, 4] and an important subset (those of complete intersection) has recently been classified by explicit construction [4].

The number of orbifolds is also growing rapidly and may offer computational advantages. They are often regarded as approximations of underlying Calabi-Yau manifolds, but the relationship remains obscure and demands attention. In particular, we would like to know to what extent such an 'approximate manifold' approximates the physics on the manifold proper. Furthermore, the even more radical suggestion that we accept singular 'manifolds' as legitimate spacetime backgrounds *per se* does not warrant any less attention.

In the following we try to elucidate these questions by examining a specific manifold as explicitly and geometrically as possible.

Although the Z-manifold [1, 5] discussed here is perhaps somewhat pathological as Calabi-Yau manifolds go, it offers the unique advantage that it can be constructed in a geometrically intuitive way. Furthermore, its alleged 'orbifold limit' is among the most popular orbifolds and has been carefully studied [2, 7].

It is also the three- (complex) dimensional analogue of K3 which is the only non-trivial Calabi-Yau manifold in two dimensions. We exploit this fact to bridge the conceptual gap between the 'lowbrow' techniques (differential geometry) employed here, and the powerful but abstract ideas of algebraic geometry.

After briefly reviewing the construction of the Z-manifold, we give explicit expressions for the metric, curvature invariants, harmonic forms and Yukawa couplings. In § 4 we define and discuss their orbifold limits and compare our results on the Z-manifold with the purely algebraic arguments of Strominger [5], and recent calculations by Dixon *et al* [6] on the orbifold. Finally we make contact with algebraic geometry and identify the physical meaning of the moduli.

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2. The Z-manifold

In the geometric construction of the Z-manifold we start with a 3-torus defined by the SU(3) root lattice and then identify points under a Z_3 group which leaves 27 points on the torus fixed [1, 5, 8]. The resulting object is singular at the fix points and is the Z-orbifold.

To obtain the Z-manifold a mathematician would now 'blow up' the fix points with P_2, the complex projective space in two dimensions.

A completely analogous construction in two dimensions using the SU(2) lattice, dividing by Z_2 and blowing up with P_1 gives K3, which became known to physicists as a gravitational instanton some years ago [9]. K3 is the only Calabi-Yau in two dimensions which is not a torus. To a physicist the 'blowing up' amounts to cutting out a ball containing the rotten point and gluing in a smooth manifold excised from an appropriate donor space. The existence of such a donor of correct type, i.e. a complex non-compact manifold which admits a Ricci-flat metric and is asymptotically Euclidean up to a discrete identification, is far from trivial and severly limits the number of Calabi-Yau manifolds that can be constructed by this cut-and-paste method.

Fortunately all such spaces have been classified in all dimensions ('spherical space forms') [10], among which we find the Eguchi-Hanson spaces (EH_n) whose boundary at infinity has the topology of S^{n-1}/Z_n that we need for our plugs ($n=2$ for K3; $n=3$ for Z).

These spaces admit one-parameter families of Ricci-flat metrics. The scale parameter (λ) controls the size of the region in which the curvature is located, and by making it sufficiently small the overlap region where the plug is glued into the hole can be made arbitrarily smooth.

The EH_n-metric can be found by making a U(n)-invariant ansatz [8, 11, 12]:

$$g_{\mu\bar{v}} = A\delta_{\mu\bar{v}} + Bz_\mu z_{\bar{v}} \tag{2.1}$$

where the z_μ are n complex coordinates. By demanding that the metric also be Kähler, flat at infinity and Ricci-flat everywhere, we find

$$\begin{aligned} A &= u^{-1}\delta^{1/n} \\ B &= -\lambda^{2n}u^{-2}\delta^{(1/n)-1} \end{aligned} \tag{2.2}$$

where $u = z_\mu z^\mu$ and

$$\delta = u^n + \lambda^{2n}. \tag{2.3}$$

This metric is singular when $u \to 0$. That this is only a coordinate singularity can be seen by studying the curvature invariants of the metric.

Using the Riemann tensors recorded in appendix 1 we find the simplest of these to be

$$R_{\mu\bar{v}\rho\bar{\sigma}}R^{\mu\bar{v}\rho\bar{\sigma}} = \frac{\Gamma(n+3)}{\Gamma(n-1)}\lambda^{2n}\delta^{-(2/n)-2} \tag{2.4}$$

which is finite for all values of u.

This can also be seen directly from the metric by changing coordinates. Following [9] the choice $y = (1/n)(z_n)^n$, $y_i = z_i/z_n$ gives

$$\begin{aligned} ds^2 = {} & |y/\lambda|^{2(n-1)}[(1+y_iy^i)\, dy\, d\bar{y} + (1+y_iy^i)^{n-1}(\bar{y}, dy\, y_i\, dy^i + y\, d\bar{y}\, y^i\, dy_i)] \\ & + \lambda^2\, ds^2(P_{n-1}) + \dots . \end{aligned} \tag{2.5}$$

In the limit $y \to 0(y_i \neq 0)$ (2.5) reduces to the second term, which is the Fubini-Study metric on P_{n-1}. Thus we have recovered the statement that these singularities should be 'blown up' with P_{n-1}, and we have every reason to believe that the cutting and pasting of metrics has produced the Z-manifold.

3. Yukawa couplings on the Z-manifold

In the heterotic string compactified on a Calabi-Yau manifold the (for us) observable part of the particle spectrum is given by the cohomology classes of forms on the manifold [1]. The only interesting forms on the Z-manifold are $(1, 1)$-forms; 9 from the torus and 27 from the blown-up fix points or, in the language of orbifoldology, 9 from the untwisted and 27 from the twisted sector. Only the twisted sector is affected by the orbifold limit so we restrict our discussion to the construction and coupling of these modes. The untwisted sector is discussed in [5].

For the purpose of calculating topological invariants like Yukawa couplings [9] we may as well choose the unique harmonic representatives of each cohomology class. An explicit representation of the harmonic $(1, 1)$-forms ω on the EH_n-plugs is easily found by noting that they are radial perturbations of the metric:

$$\omega_{\mu\bar{\nu}} = \tfrac{1}{2}\delta_u g_{\mu\bar{\nu}} = z^\lambda \partial_\lambda g_{\mu\bar{\nu}} + z^{\bar{\lambda}} \partial_{\bar{\lambda}} g_{\mu\bar{\nu}}. \tag{3.1}$$

This is proved in appendix 2 by a reasoning similar to that used in constructing the EH_n-metric.

From (2.1) we find

$$\omega_{\mu\bar{\nu}} = C\delta_{\mu\bar{\nu}} + Dz_\mu z_{\bar{\nu}} \tag{3.2}$$

with

$$\begin{aligned} C &= -\lambda^{2n} u^{-1} \delta^{(1/n)-1} \\ D &= \lambda^{2n} u^{-2} \delta^{(1/n)-2}(\lambda^{2n} + nu^n). \end{aligned} \tag{3.3}$$

Notice that these modes fall off as u^{-n} at radial infinity. The form is therefore localised at the origin ($u = 0$) so we can approximate the twisted forms on the Z-manifold by these forms. For the same reason overlap integrals only have support on the EH_n plug. Notice also that the normalisation of these modes is fixed by the normalisation of the metric (2.1).

In order to discuss the orbifold limit we need to know the value of the inner product (ω, ω), which can be written [13]

$$(\omega, \omega) = (-1)^{n(n-1)/2}(2i)^n \int \omega_{\mu\bar{\nu}} \omega^{\mu\bar{\nu}} g \, \mathrm{d}^n z \, \mathrm{d}^n \bar{z} \tag{3.4}$$

where $g = \det g_{\mu\bar{\nu}}$ is the square root of the determinant of the real metric. Raising indices with the inverse metric we find

$$\omega_{\mu\bar{\nu}} \omega^{\mu\bar{\nu}} = n(n-1)\lambda^{4n}\delta^{-2} \tag{3.5}$$

which depends only on the radial coordinate $u = r^2$. The measure in polar coordinates is given by

$$i^n(-1)^{n(n-1)/2} \, \mathrm{d}^n z \, \mathrm{d}^n \bar{z} = 2^{n-1} u^{n-1} \, \mathrm{d}u \, \mathrm{d}^{2n-1}\Omega \tag{3.6}$$

and we find for $n=2$(K3) and $n=3(Z)$:

$$(\omega, \omega)=\Lambda^n/n \tag{3.7}$$

where

$$\Lambda=-2\pi \mathrm{i}\lambda^2. \tag{3.8}$$

The Yukawa coupling (κ_n) is defined by

$$\kappa_n=\int \omega \wedge \omega \wedge \ldots \wedge \omega=\int \omega^n \tag{3.9}$$

with ω given by (3.2). Because of the antisymmetry of the wedge product only factors at most quadratic in z_μ can appear, and we obtain

$$\kappa_n=(-1)^{n(n-1)/2}n!\int (C^n+C^{n-1}Du)\,\mathrm{d}^n z\,\mathrm{d}^n \bar{z}. \tag{3.10}$$

Inserting C and D from (3.3) and integrating gives

$$\kappa_n=(-1)^{n+1}\frac{\Lambda^n}{n}=(-1)^{n+1}(\omega, \omega). \tag{3.11}$$

4. The orbifold limit

By definition, in the 'orbifold limit' we should recover the flat torus (mod Z_3) with 27 isolated conical sigularities.

Consider first the curvature invariant (2.4). For any finite $u \neq 0$ we see that $(RIEM)^2$ vanishes when $\lambda \to 0$, while at $u=0$ it explodes in the same limit; i.e. in the limit $\lambda \to 0$ the curvature vanishes outside $u=0$, while it goes singular at this point.

The same result is obtained by studying the metric. From (2.1), valid for $u \neq 0$, we see that the metric goes flat when $\lambda \to 0$. Near $u=0$ we switch to (2.5), which shows that when $\lambda \to 0$ for any finite y the P_{n-1}-metric is turned off and the EH_n-metric explodes.

Obviously, $\lambda \to 0$ is the orbifold limit.

Next notice that in the orthogonal basis automatically implied by the EH_n-approximation both the normalisation matrix (3.7) and the Yukawa coupling (3.11) vanish in the orbifold limit.

These results appear inconsistent with those of [6], where this coupling was calculated exactly in the orbifold limit of the string. Omitting the non-perturbative contributions, their result for the coupling of the twisted modes is also finite, $\kappa=(2\pi)^6 3^{-9/4}\Gamma^{-1}(\frac{1}{3})$, but not zero.

If, however, we insist on canonical normalisation of the kinetic term, then we must be more careful. Canonical normalisation is achieved by absorbing (ω, ω) into the four-dimensional fields. This field redefinition also changes the effective Yukawa coupling, which now becomes

$$\tilde{\kappa}_3=(\omega, \omega)^{-1/2}. \tag{4.1}$$

From (3.7) and (3.8) we see that this quantity diverges in the orbifold limit.

This is in agreement with [5], where a very abstract argument led to the conclusion that Yukawa couplings diverge while the normalisation vanishes in the limit where some of the moduli of the Z-manifold are sent to zero. It only remains to argue that the scale parameters on the EH_n-plugs are moduli on the manifold.

This would be easy if we could represent the manifold algebraically as a polynomial constraint in projective space, since then the moduli are simply coefficients in the polynomial. By Kodaira's embedding theorem it is true that all Calabi-Yau manifolds are projective algebraic (because $h^{20}=0$) but the algebraic representation of the Z-manifold is not known at present. Also, it cannot be represented by complete intersection which would have been an enormous simplification. This follows from the fact that complete intersection Calabi-Yau manifolds all have negative Euler number [4] while the Euler number of Z is 72.

However, strong circumstantial evidence can be found by studying K3, which not only has the geometric construction given above, but can also be represented as the complete intersection of a quartic in P_3. For definiteness consider the ground state

$$p=\sum_{A=1}^{4} z_A^4=0 \tag{4.2}$$

which is smooth because $dp \neq 0$ when $p=0$.

The argument relies heavily on a remarkable observation due to Kodaira [14]. He noted that the space of deformations of the complex structure on the manifold, which is the space of tangent bundle valued 1-forms $H^1(M, T)$, is at least partly parametrised by the deformations of the defining polynomial p. We can for example regard the monomials missing from p in (4.2) as spanning this space. Furthermore, all manifolds with vanishing first Chern class possess a unique holomorphic n-form ('holomorphic volume form') which can be used to trade a tangent space index for $n-1$ cotangent space indices. In other words we have the bundle isomorphism $T \simeq \Omega^{n-1,0}$ which induces the group isomorphism

$$H^1(M, T) \simeq H^1(M, \Omega^{n-1,0}) \simeq H^{n-1,1}(M) \tag{4.3}$$

where the last connection to ordinary Dolbeault cohomology follows by Serre duality.

Hence in two dimensions (K3) the polynomial deformations parametrise the (1, 1)-forms, while in three dimensions they parametrise the (2, 1)-forms. This point of view was pursued vigorously in [15] to compute the Yukawa couplings on certain three-dimensional complete intersection Calabi-Yau manifolds. This reference contains a beautifully intuitive description of the polynomial deformation method in terms of lapse and shift functions which should be familiar from general relativity.

We can therefore to some extent count the number of harmonic forms by counting coefficients in the most general polynomial of the required degree, in our case quartics. Not all deformations change the complex structure however. Any deformation proportional to $\partial_A p$ is simply a change of coordinates. This leaves 19 non-trivial deformations of p corresponding to 19 of the 20 (1, 1)-forms on K3. Sixteen of these come from blowing up fix points.

There is a simple way to make this manifold singular. Whenever $p=q^2$, where q is some quadric, the manifold is singular because $q=0$ satisfies both $p=0$ and $dp=0$.

If we conjecture that this is the orbifold limit of K3, and also believe that the 16 scale parameters on the 16 EH_2-plugs used to blow up the fix points on the K3 orbifold are moduli, then there should be a sixteen-parameter family of quartic polynomials such that when any coefficient vanishes the quartic degenerates to the square of a quadric. In other words, how many different ways can we turn off four of the coefficients in the general quartic and still 'complete the square' using only trivial deformations (i.e. polynomials in the ideal generated by $\partial_A p$)? The answer is exactly sixteen.

We therefore do believe that both the 'moduli limit' and the 'λ limit' give the Z-orbifold, and we conclude that our geometric arguments are in agreement with the abstract algebraic treatment given in [5].

We have not, however, succeeded in explaining why the conformal field theory calculation on the flat orbifold seems to disagree with the orbifold limit of the result obtained on the smooth manifold. It would be of great interest to study this problem on a three-dimensional complete intersection Calabi–Yau manifold like, e.g., a quintic in P_4, where explicit algebraic representations of all the zero modes are known. The Yukawa couplings are in this case easy to compute on the manifold, and an orbifold limit along the lines sketched above for K3 should be feasible. Furthermore, the conformal field theory of this model has recently been constructed [16].

In conclusion, the 'orbifold limit' remains obscure, but it seems likely that studying the conformal field theories of complete intersection Calabi–Yau manifolds will shed some light on this important question.

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Appendix 1. Riemann tensors on EH_n

We record here the Riemann tensors on EH_n which are needed for calculating curvature invariants.

The Riemann tensor simplifies greatly on a complex manifold:

$$R_{\mu\bar{\nu}\rho\bar{\sigma}} = \partial_\rho \partial_{\bar{\sigma}} g_{\mu\bar{\nu}} - g^{\alpha\bar{\beta}} \partial_\rho g_{\mu\bar{\beta}} \partial_{\bar{\sigma}} g_{\alpha\bar{\nu}}.$$

In particular, on EH_n the inverse metric is

$$g^{\mu\bar{\nu}} = a\delta^{\mu\bar{\nu}} + bz^\mu z^{\bar{\nu}}$$

with

$$a = A^{-1}$$
$$b = -BA^{-1}(A+Bu)^{-1}$$

where A and B are given in (2.2). We find

$$R_{\mu\bar{\nu}\rho\bar{\sigma}} = rI_{\mu\bar{\nu}\rho\bar{\sigma}} + sX_{\mu\bar{\nu}\rho\bar{\sigma}} + tY_{\mu\bar{\nu}\rho\bar{\sigma}}$$

where

$$I_{\mu\bar{\nu}\rho\bar{\sigma}} = \delta_{\mu\bar{\nu}}\delta_{\rho\bar{\sigma}} + \delta_{\mu\bar{\sigma}}\delta_{\rho\bar{\nu}}$$
$$X_{\mu\bar{\nu}\rho\bar{\sigma}} = \delta_{\mu\bar{\nu}} z_\rho z_{\bar{\sigma}} + \delta_{\mu\bar{\sigma}} z_\rho z_{\bar{\nu}} + \delta_{\rho\bar{\nu}} z_\mu z_{\bar{\sigma}} + \delta_{\rho\bar{\sigma}} z_\mu z_{\bar{\nu}}$$
$$Y_{\mu\bar{\nu}\rho\bar{\sigma}} = z_\mu z_{\bar{\nu}} z_\rho z_{\bar{\sigma}}$$
$$r = -\lambda^{2n} u^{-2} \delta^{(1/n)-1}$$
$$s = \lambda^{2n} u^{-3} \delta^{(1/n)-2}[(n+1)u^n + \lambda^{2n}]$$
$$t = -\lambda^{2n} u^{-4} \delta^{(1/n)-3}[(n+1)(n+2)u^{2n} + 4(n+1)u^n\lambda^{2n} + 2\lambda^{4n}].$$

λ is the scale parameter and $\delta = u^n + \lambda^{2n}$.

To find the contravariant Riemann tensor

$$R^{\mu\bar{\nu}\rho\bar{\sigma}} = RI^{\mu\bar{\nu}\rho\bar{\sigma}} + SX^{\mu\bar{\nu}\rho\bar{\sigma}} + TY^{\mu\bar{\nu}\rho\bar{\sigma}}$$

it is easiest to observe that only the $\delta^{\mu\bar{\nu}}$-piece of $g^{\mu\bar{\nu}}$ can contribute to R, which must therefore be ra^4, and use the Ricci flatness to derive S and T. We find

$$R = -\lambda^{2n}u^2\delta^{-(3/n)-1}$$

$$S = \lambda^{2n}u^{1-n}\delta^{-(3/n)-1}[(n+1)u^n + n\lambda^{2n}]$$

$$T = -\lambda^{2n}u^{-2n}\delta^{-(3/n)-1}[(n+1)(n+2)u^{2n} + 2n(n+1)\lambda^{2n}u^n + n(n-1)\lambda^{4n}].$$

Appendix 2. Harmonic forms on EH_n

We show here that the radial perturbations (3.2) of the metric (2.1) are the harmonic (1, 1)-forms on EH_n.

The (1, 1) form $\omega = \omega_{\mu\bar{\nu}}\mathrm{d}z^\mu \wedge \mathrm{d}z^{\bar{\nu}}$ is harmonic ($\Delta\omega = 0$) iff it is curl free ($\mathrm{d}\omega = 0$) and divergence free ($\mathrm{d}^\dagger\omega = 0$). The last condition is not easy to implement so we make the observation that it is sufficient that the (1, 1)-form be trace and curl free for its divergence to vanish. This is easily established by taking the covariant derivative of the trace-free condition

$$g^{\mu\bar{\nu}}\omega_{\mu\bar{\nu}} = 0$$

and using that the metric is covariantly constant. The curl-free condition is

$$\nabla_{[\rho}\omega_{\mu]\bar{\nu}} = \partial_{[\rho}\omega_{\mu]\bar{\nu}} = 0.$$

Again making a U(n)-invariant ansatz as we did when deriving the metric

$$\omega_{\mu\bar{\nu}} = \alpha\delta_{\mu\bar{\nu}} + \beta z_\mu z_{\bar{\nu}}$$

and using the two equations above we find

$$\alpha = Nu^{-1}\delta^{(1/n)-1}$$

$$\beta = -Nu^{-2}\delta^{(1/n)-2}(\lambda^{2n} + nu^n).$$

With the normalisation $N = -\lambda^{2n}$ we see that this is precisely the metric perturbation (3.2).

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